

# **Hierarchical Models for Combining Information in Interlaboratory Studies**

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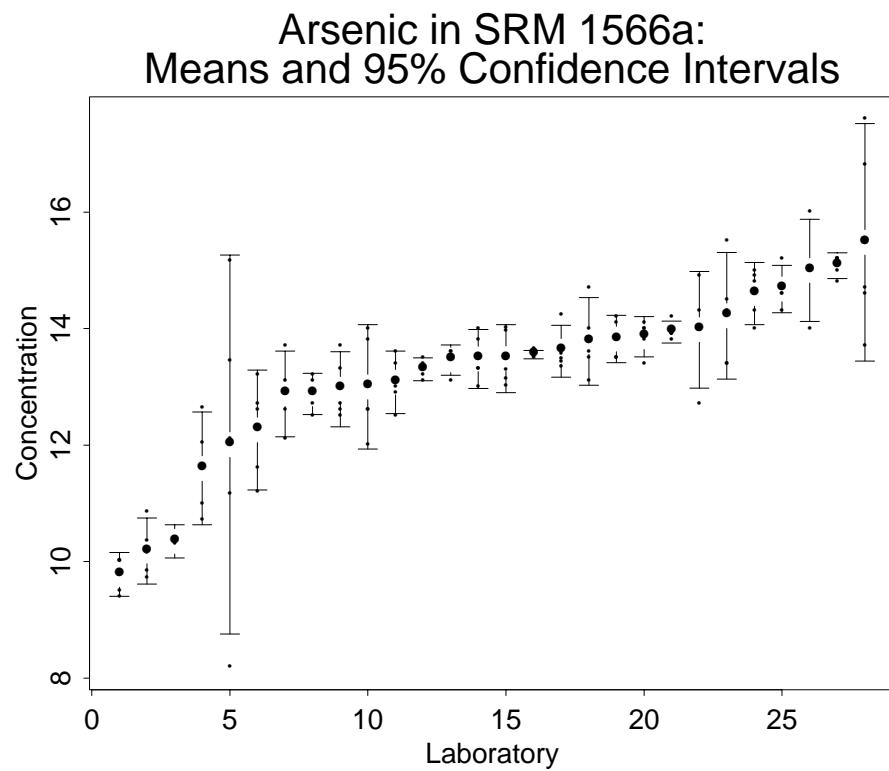
## The Scenario

- The  $i$ th of  $p$  laboratories makes  $n_i$  repeated measurements of the same quantity.
- The laboratories make measurements with different precisions.
- The selected laboratories can be regarded as a random sample from a population (i.e., they're exchangeable).

## The Problem

**How should one estimate the ‘grand mean’ and between-laboratory variance?**

## **Example: Arsenic in Oyster Tissue (NIST Standard Reference Material 1566a)**



# Hierarchical Model With Noninformative Priors (Model A)

$i = 1, \dots, p$  indexes laboratories

$j = 1, \dots, n_i$  indexes measurements

$$p(x_{ij}|\theta_i, \sigma_i^2) = N(\theta_i, \sigma_i^2)$$

$$p(\sigma_i) \propto 1/\sigma_i$$

$$p(\theta_i|\mu, \sigma^2) = N(\mu, \sigma^2)$$

$$p(\mu) = 1$$

$$p(\sigma) = 1$$

## A Useful Probability Density

Let  $T_\nu$  and  $Z$  denote independent Student- $t$  and standard normal random variables, and assume that  $\gamma \geq 0$  and  $\nu > 0$ . Then

$$U = T_\nu + Z\sqrt{\frac{\gamma}{2}}$$

has density

$$f_\nu(u; \theta) \equiv \frac{1}{\nu/2\sqrt{\pi}} \int_0^\infty \frac{y^{(\nu+1)/2-1} e^{-y\left[1+\frac{u^2}{\gamma y + \nu}\right]}}{\sqrt{\gamma y + \nu}} dy.$$

## Posterior of $(\mu, \sigma)$ : Hierarchical Model A

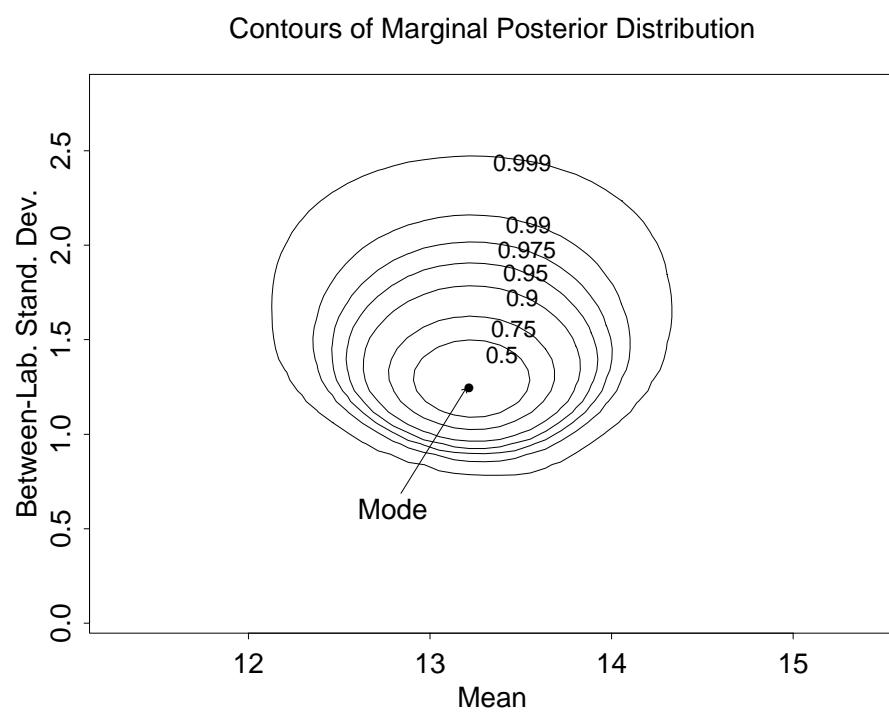
- The posterior of  $(\mu, \sigma)$  for general  $p(\sigma)$  is

$$p(\mu, \sigma | \{x_{ij}\}) \propto p(\sigma) \prod_{i=1}^p \frac{1}{t_i} f_{n_i-1} \left[ \frac{x_i - \mu}{t_i}; \frac{2\sigma^2}{t_i^2} \right].$$

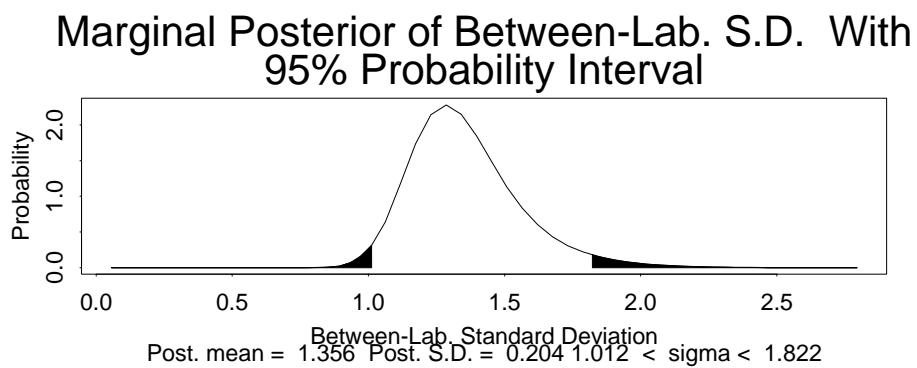
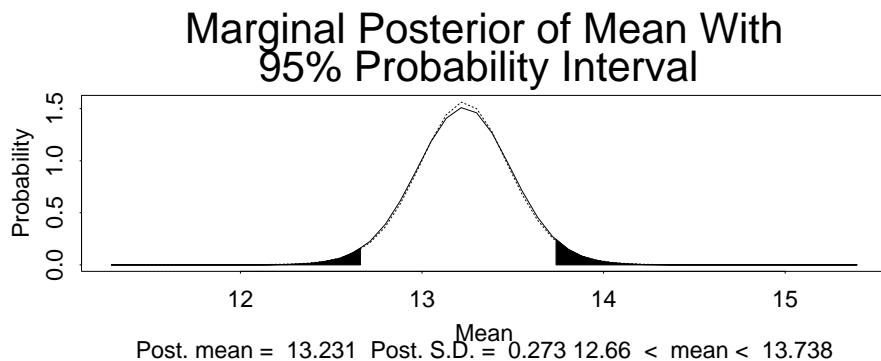
- The posterior of  $\mu$  given  $\sigma = 0$  is a product of scaled  $t$ -densities centered at the  $x_i$ .
- We will take  $p(\sigma) = 1$ .

# Model A Posteriors for Arsenic Data:

$(\mu, \sigma)$



# Model A Posteriors for Arsenic Data: $\mu$ and $\sigma$



# Small Simulation Comparing Bayesian and Frequentist Intervals

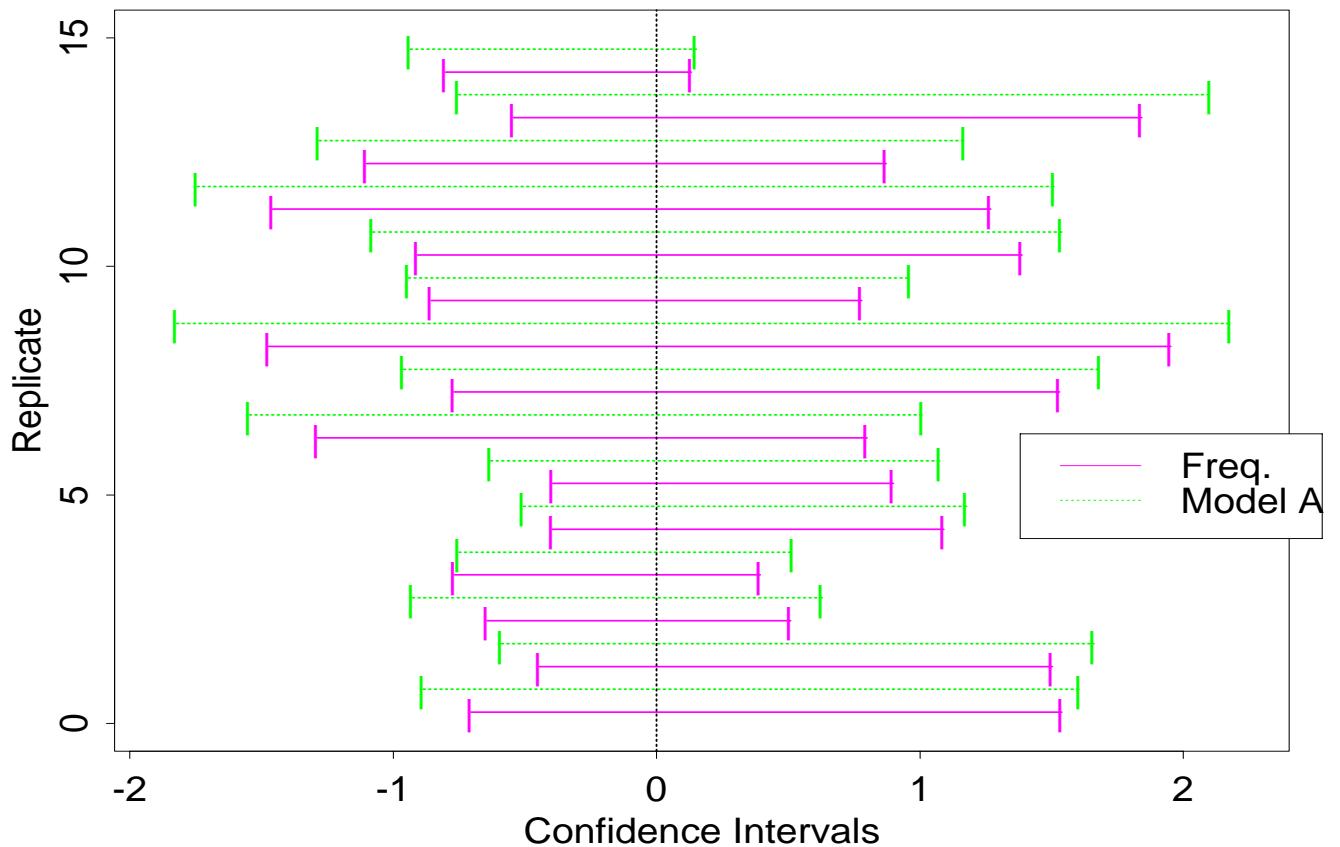
$$\mu = 0$$

$$\sigma_i = \sigma_e$$

$$\sigma^2 + \sigma_e^2 = 1$$

$$\rho = \sigma^2 / (\sigma_e^2 + \sigma^2) = 1/2$$

## Comparison of Intervals on the Mean



## A Second Hierarchical Model (Model B)

$i = 1, \dots, p$  indexes laboratories

$j = 1, \dots, n_i$  indexes measurements

$$\begin{aligned} p(x_{ij} | \theta_i, \sigma_i^2) &= N(\theta_i, \sigma_i^2) \\ p(\sigma_i^2) &= ,^{-1}(\lambda_2, \alpha^2) \\ p(\theta_i | \mu, \sigma^2) &= N(\mu, \sigma^2) \\ p(\mu) &= 1 \\ p(\sigma^2) &= ,^{-1}(\lambda_1, \alpha^2) \\ p(\alpha^2) &\propto 1/\alpha^2 \end{aligned}$$

## Some Consequences of Model B

Denote intralab correlations by

$$\rho_i = \frac{\sigma^2}{\sigma^2 + \sigma_i^2}.$$

Then

$$p(\rho_i | \alpha^2) = \beta(\lambda_2, \lambda_1),$$

for all  $i$  and all  $\alpha^2$ . So the constants  $(\lambda_1, \lambda_2)$  summarize prior information on the  $\rho_i$ . For  $\lambda_1 = \lambda_2 = 1$ ,  $p(\rho_i) = U(0, 1)$ .

## Posterior of $(\mu, \sigma)$ : Hierarchical Model B

$$p(\mu, \sigma^2, \alpha^2 | \{x_{ij}\}) \propto \frac{e^{-\alpha^2/\sigma^2} (\alpha^2)^{p\lambda_2 + \lambda_1}}{(\sigma^2)^{\lambda_1 + 1}} p(\alpha^2) \\ \cdot \left[ \prod_{i=1}^p \left( \nu_i t_i^2 + 2\alpha^2/n_i \right)^{-(\eta_i+1)/2} \right] \left[ \prod_{i=1}^p f_{\eta_i}(x, \varphi_i) \right],$$

where  $\eta_i = n_i - 2\lambda_2 - 1$ ,

$$\varphi_i = \left( \frac{\eta_i}{\nu_i} \right) \frac{2\sigma^2}{t_i^2 + 2\alpha^2/(n_i \nu_i)},$$

$$x = \frac{(x_i - \mu)\sqrt{\eta_i}}{\sqrt{\nu_i t_i^2 + 2\alpha^2/n_i}}$$

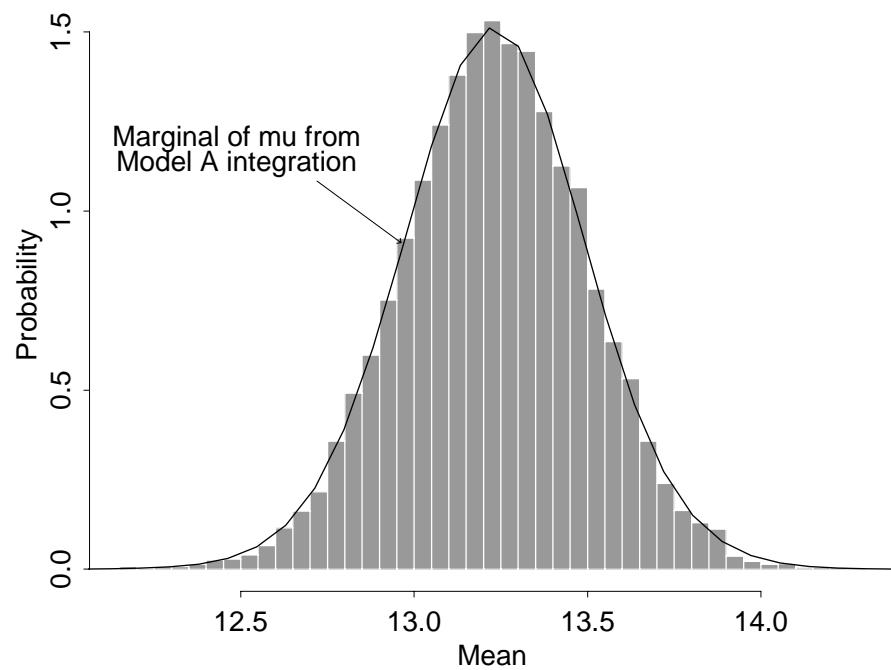
For  $n_i$  sufficiently large, Model B posterior approaches the result for Model A.

## BUGS Code for Model B Posterior Calculations

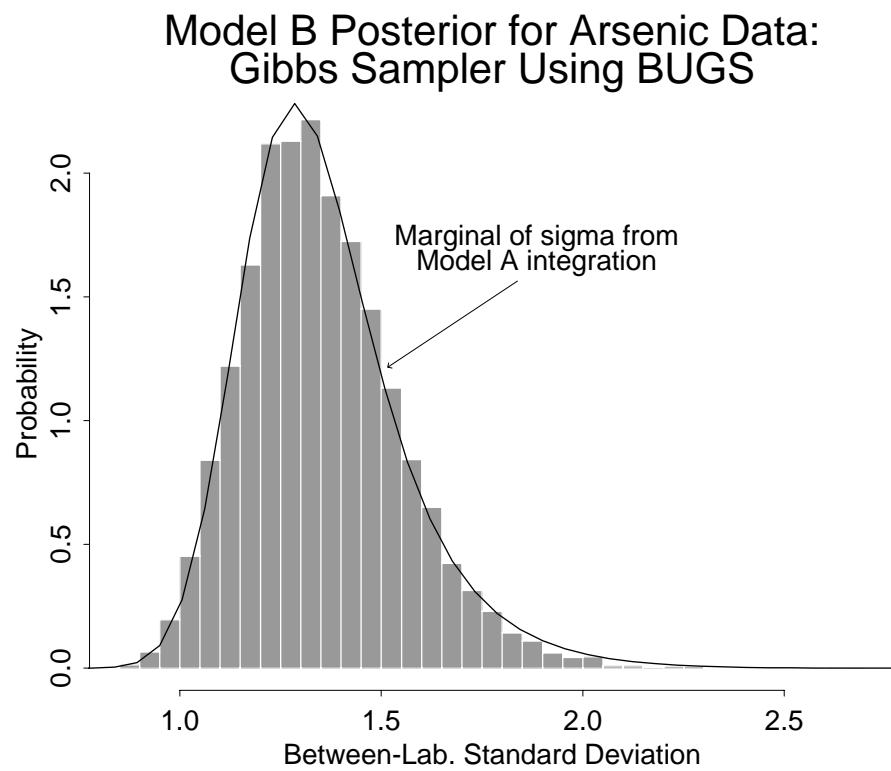
```
model arsenic;
const
  SAMPLES  = 137,    # Number of data values
  LABS     = 28,      # Number of labs
  par1     = 1,       # "Between" Shape
  par2     = 1;       # "Within" Shape
var
  y[SAMPLES], lab[SAMPLES], delta[LABS], mu,
  tau.within[LABS], rho[LABS],
  sig.within[LABS], tau.between,
  sig.between, alpha;
data lab,y in "arsenic.dat";
inits in "arsenic.in";
{
for (i in 1:SAMPLES) {
  y[i] ~ dnorm(delta[lab[i]], tau.within[lab[i]]);
}
for (i in 1:LABS)  {
  delta[i] ~ dnorm(mu, tau.between);
  tau.within[i] ~ dgamma(par2, alpha)
  sig.within[i] <- sqrt(1/tau.within[i])
  rho[i]       <- tau.within[i]/(tau.between+tau.within[i])
}
mu           ~ dnorm(0.0, 1.0E-10);
alpha         ~ dgamma(0.001, 0.001);
tau.between ~ dgamma(par1, alpha);
sig.between <-sqrt(1/tau.between)
}
```

## Posterior Distribution of $\mu$ for the Arsenic Data (Model B)

Model B Posterior for Arsenic Data:  
Gibbs Sampler Using BUGS



## Posterior Distribution of $\sigma$ for the Arsenic Data (Model B)

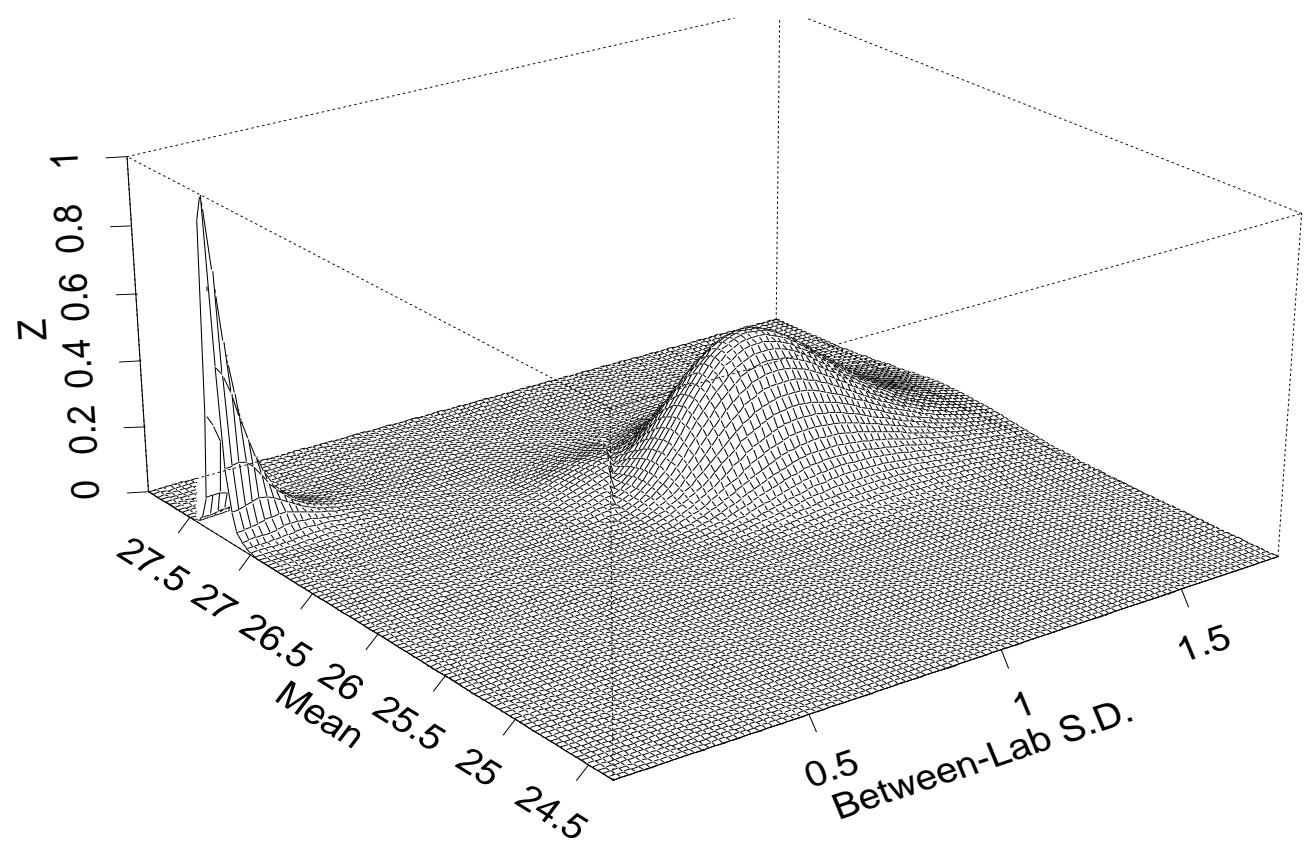


**Example: Dietary Fiber in Apricots**  
**Li and Cardozo (1994)**  
**J. Of AOAC Int., 77, p. 689**

Lab.	$x_i$	$s_i^2$	$n_i$
1	25.32	0.37	2
2	26.72	0.62	2
3	27.89	0.35	2
4	27.70	1.85	2
5	27.42	0.61	2
6	24.30	0.21	2
7	27.11	0.37	2
8	27.28	0.09	2
9	25.37	0.08	2

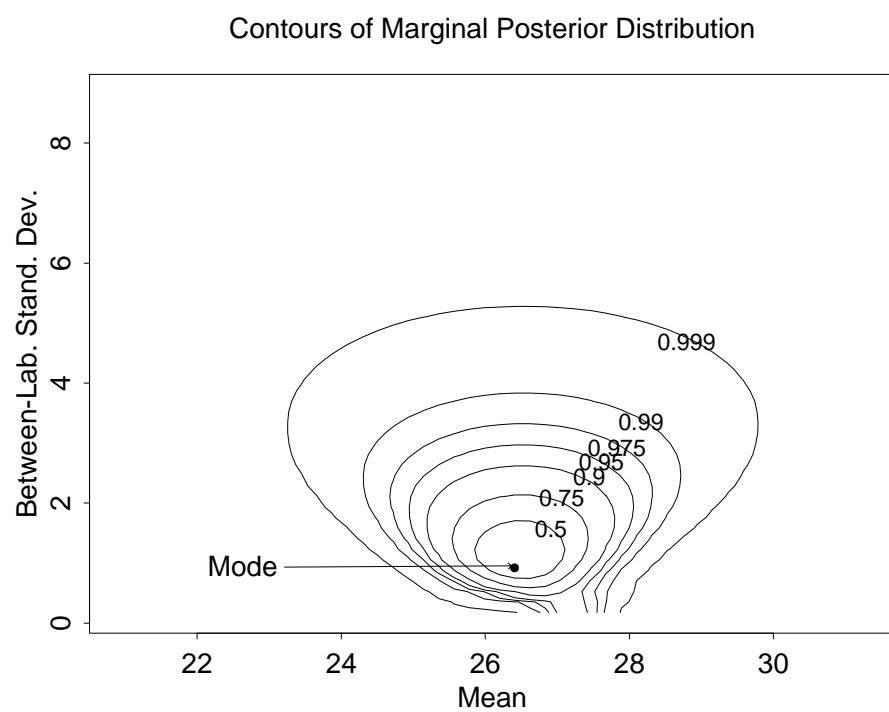


## Profile Likelihood for Apricot Fiber Data

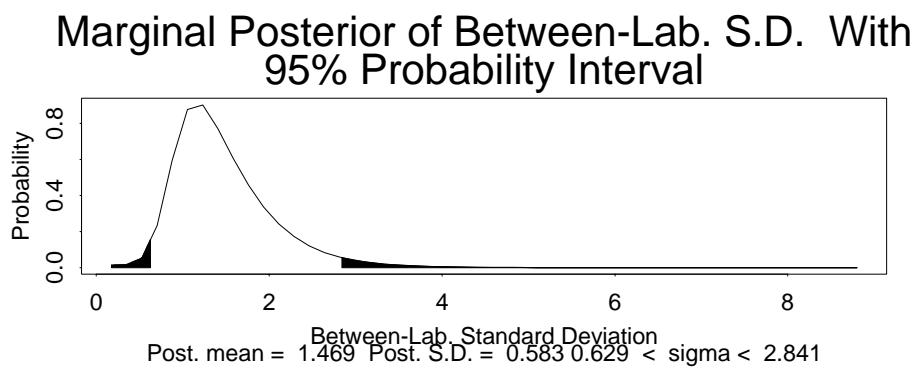
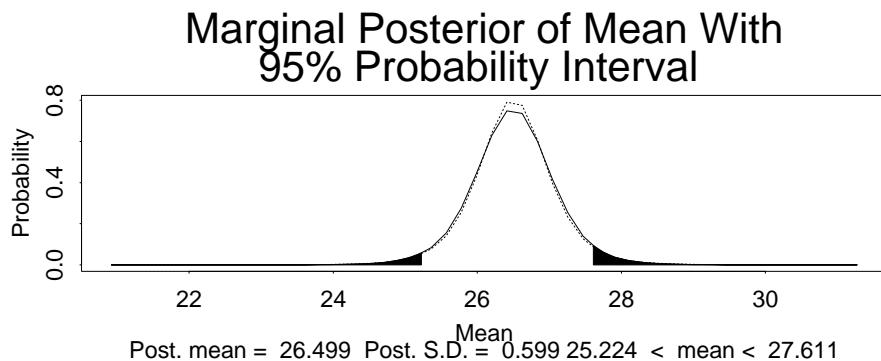


# Model A Posteriors for Apricot Data:

$(\mu, \sigma)$

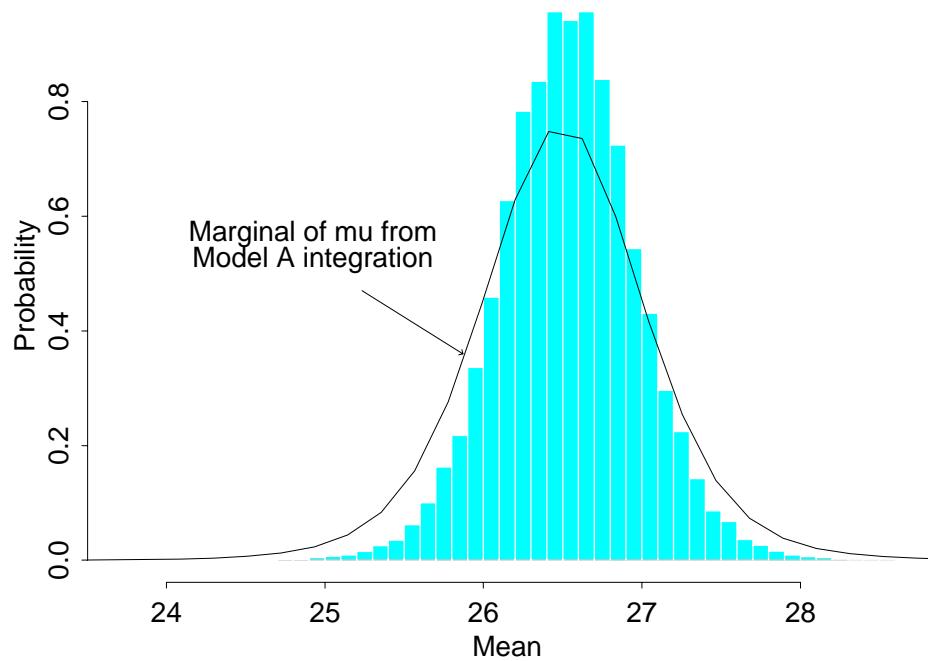


# Model A Posteriors for Apricot Data: $\mu$ and $\sigma$

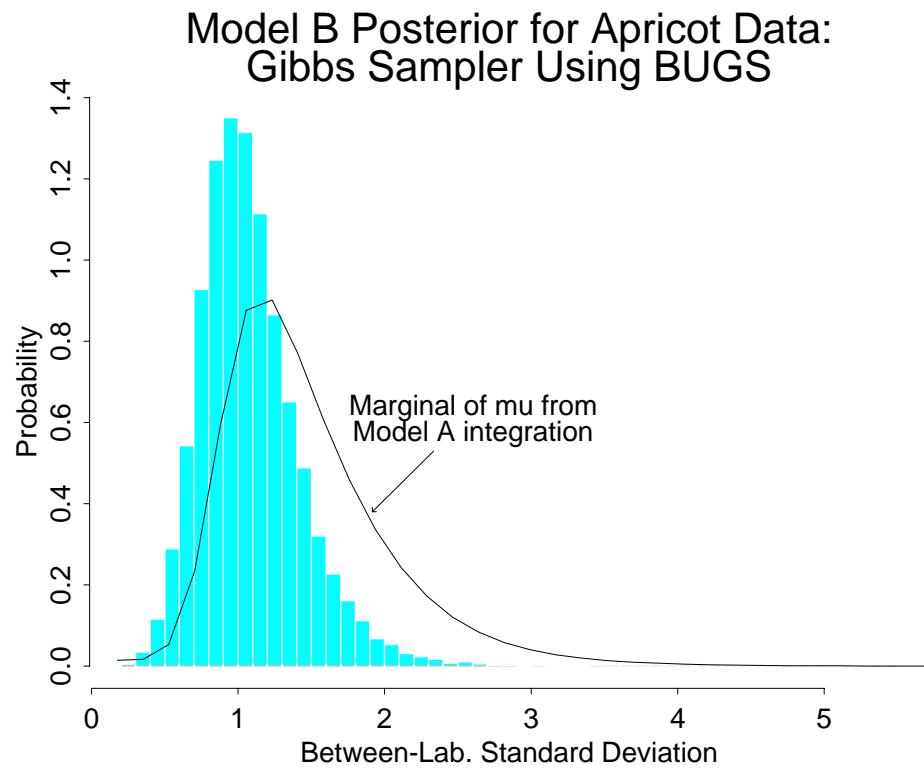


## Posterior Distribution of $\mu$ for the Apricot Data (Model B)

Model B Posterior for Apricot Data:  
Gibbs Sampler Using BUGS



## Posterior Distribution of $\sigma$ for the Apricot Data (Model B)



## **Summary and Conclusions**

- Two hierarchical models are proposed for use in combining information in ‘one-way models’; specifically for the grand mean and inter-laboratory variance in collaborative studies.
- The models allow for heteroscedasticity and unbalancedness.
- The models give similar results for  $n_i$  ‘large enough’.
- A simulation suggests that at least Model A leads to probability regions with reasonable frequency properties.